The volume growth of hyperkähler manifolds of type A_∞

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Abstract

We study the volume growth of hyperkähler manifolds of type A_∞ constructed by Anderson-Kronheimer-LeBrun [1] and Goto [10]. These are noncompact complete 4-dimensional hyperkähler manifolds of infinite topological type. These manifolds have the same topology but the hyperkähler metrics are depends on the choice of parameters. By taking a certain parameter, we show that there exists a hyperkähler manifold of type A_∞ whose volume growth is r^α for each $3 < \alpha < 4$.

1 Introduction

A hyperkähler manifold is, by definition, a Riemannian manifold (X, g) of real dimension $4n$ equipped with three complex structures I_1, I_2, I_3 satisfying the quaternionic relations $I_1^2 = I_2^2 = I_3^2 = I_1 I_2 I_3 = -\mathrm{id}$ with respect to all of which the metric g is Kählerian. Then the holonomy group of g is a subgroup of $Sp(n)$ and g is Ricci-flat.

In this paper we focus on the volume growth of hyperkähler metrics. The notion of the volume growth are considered for a Riemannian manifold (X, g). We denote by $V_g(p_0, r)$ the volume of the ball $B_g(p_0, r) \subset X$ of radius r centered at $p_0 \in X$. Then we say that the volume growth of g is $f(r)$ if the condition

$$0 < \liminf_{r\to+\infty} \frac{V_g(p_0, r)}{f(r)} \le \limsup_{r\to+\infty} \frac{V_g(p_0, r)}{f(r)} < +\infty.$$

holds for some $p_0 \in X$. From Bishop-Gromov comparison theorem [6][12], the above condition is independent of $p_0 \in X$ if g has the nonnegative Ricci

curvature.
For instance, the volume growth of Euclidean space $\mathbb{R}^4$ is r^4 and the volume growth of $\mathbb{R}^3 \times S^1$ with the flat metric is r^3. These are trivial examples and there are also nontrivial examples such as ALE hyperkähler metrics constructed in [8][9][15] whose volume growth is r^4, and multi-Taub-NUT metrics [13][16][18] whose volume growth is r^3.
Thus there are several examples of complete hyperkähler manifolds whose volume growth is r^k for positive integers k. On the other hand, there exist complete Ricci-flat Kähler manifolds of complex dimension n whose volume growth are $r^{(2n)/(n+1)}$ [2][3][19].
In this paper we will show that there is a family of complete hyperkähler manifolds of real dimension 4 whose volume growth is less than r^4 and more than r^3.
Anderson, Kronheimer and LeBrun constructed the hyperkähler manifolds of type A_∞ by Gibbons Hawking ansatz in [1], which are 4-dimensional non-compact complete hyperkähler manifolds of infinite topological type. Here infinite topological type means that the homology groups are infinitely generated. The same metrics were constructed by hyperkähler quotinet method due to Goto in [10]. Each of the metrics in [1] is constructed from an element of

$$(\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}} := \{\lambda = (\lambda_n)_{n\in\mathbb{Z}} \in (\mathrm{Im}\mathbb{H})^{\mathbb{Z}};\ \sum_{n\in\mathbb{Z}} \frac{1}{1+|\lambda_n|} < +\infty\},$$

where $\mathbb{H}$ is the quaternions and $\mathrm{Im}\mathbb{H}$ the 3-dimensional subspace formed by the purely imaginary quaternions. We denote by (X_λ, g_λ) the hyperkähler metric of type A_∞ constructed from $\lambda = (\lambda_n)_{n\in\mathbb{Z}} \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$. The purpose of this paper is studying the asymptotic behavior of $V_{g_\lambda}(p_0, r)$ for some $p_0 \in X_\lambda$, and observe how the volume growth of g_λ depends on the choice of λ. The main result is described as follows.

Theorem 1.1. *For each $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$ and $p_0 \in X_\lambda$, the function $V_{g_\lambda}(p_0, r)$ satisfies*

$$0 < \liminf_{r\to+\infty} \frac{V_{g_\lambda}(p_0, r)}{r^2\tau_\lambda^{-1}(r^2)} \le \limsup_{r\to+\infty} \frac{V_{g_\lambda}(p_0, r)}{r^2\tau_\lambda^{-1}(r^2)} < +\infty,$$

where the fuction $\tau_\lambda : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is defined by

$$\tau_\lambda(R) := \sum_{n\in\mathbb{Z}} \frac{R^2}{R+|\lambda_n|}$$

for $R \ge 0$.

Applying Theorem 1.1 to some $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$, we can find hyperkäler manifolds whose volume growth is given as follows.

Theorem 1.2. (1) *There is a hyperkähler manifold* (X_λ, g_λ) *for each* $3 < \alpha < 4$ *which satisfies*

$$0 < \liminf_{r \to +\infty} \frac{V_{g_\lambda}(p_0, r)}{r^\alpha} \leq \limsup_{r \to +\infty} \frac{V_{g_\lambda}(p_0, r)}{r^\alpha} < +\infty.$$

(2) *There is a hyperkähler manifold* (X_λ, g_λ) *which satisfies*

$$\lim_{r \to +\infty} \frac{V_{g_\lambda}(r)}{r^4} = 0, \quad \lim_{r \to +\infty} \frac{V_{g_\lambda}(r)}{r^\alpha} = +\infty$$

for any $\alpha < 4$.

Next we denote by $g_\lambda^{(s)}$ the Taub-NUT deformation of g_λ where $s > 0$ is the parameter of deformations. Then the volume growth of $g_\lambda^{(s)}$ is given by the following.

Theorem 1.3. *Let* $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$ *and* $s > 0$. *Then the volume growth of hyperkähler metric* $g_\lambda^{(s)}$ *satisfies*

$$\lim_{r \to +\infty} \frac{V_{g_\lambda^{(s)}}(r)}{r^3} = \frac{8\pi^2}{3\sqrt{s}}.$$

In Section 2, we review the construction and the basic properties of hyperkähler manifolds of type A_∞. Although there are two constructions by Gibbons-Hawking ansatz and hyperkähler quotient method, we adopt the latter way according to [10] since the argument in Section 3 is due to Goto's construction. Then we obtain the hyperkähler manifold (X_λ, g_λ) as a hyperkähler quotient, and there are an S^1-action on X_λ preserving the hyperkähler structure and the hyperkähler moment map $\mu_\lambda : X_\lambda \to \mathrm{Im}\mathbb{H}$.
For the proof of Theorem 1.1, we need upper and lower bounds for the function $V_{g_\lambda}(p_0, r)$, which are discussed in Sections 3 and 4, respectively. In Section 3 the upper bound for $V_{g_\lambda}(p_0, r)$ will be obtained as follows. We fix $p_0 \in X_\lambda$ to be $\mu_\lambda(p_0) = 0 \in \mathrm{Im}\mathbb{H}$. Then we obtain two inequalities

$$\begin{aligned} \mathrm{vol}_{g_\lambda}(\mu_\lambda^{-1}(B_R)) &\leq P_+ R^2 \varphi_\lambda(R), \\ d_{g_\lambda}(p_0, p)^2 &\geq Q_- |\mu_\lambda(p)| \cdot \varphi_\lambda(|\mu_\lambda(p)|), \end{aligned}$$

for each $p \in X_\lambda$, where vol_{g_λ} is the Riemannian measure, d_{g_λ} is the Riemannian distance, $B_R := \{\zeta \in \mathrm{Im}\mathbb{H};\ |\zeta| < R\}$, and P_+, Q_- are positive constants.

The former inequality is obtained from the hyperkähler construction, and the latter one is obtained from Gibbons Hawking ansatz. By combining these inequalities we have the upper bound for $V_{g_\lambda}(p_0, r)$.
Next we discuss the lower bound for $V_{g_\lambda}(p_0, r)$ in Section 4. If we try to bound the function $V_{g_\lambda}(p_0, r)$ from below in a similar method as in Section 3, then we need to show the inequality

$$d_{g_\lambda}(p_0, p)^2 \quad \leq \quad Q_+|\mu_\lambda(p)| \cdot \varphi_\lambda(|\mu_\lambda(p)|),$$

which seems to be hard to show for any $p \in X_\lambda$. But it does not mean that the author can give a counterexample to the inequality. In Section 4, we take an open subset of $B_{g_\lambda}(p_0, r)$ which consists of points $p \in B_{g_\lambda}(p_0, r)$ satisfying the inequality $d_{g_\lambda}(p_0, p)^2 \leq Q_+|\mu_\lambda(p)| \cdot \varphi_\lambda(|\mu_\lambda(p)|)$, and compute the volume of the subset, which is the lower bound for $V_{g_\lambda}(p_0, r)$.
From two types of estimates obtained in Sections 3 and 4, we prove Theorem 1.1 in Section 5.
We compute the volume growth of some examples concretely in Section 6, and obtain Theorem 1.2. Section 7 is devoted to studying the volume growth of the Taub-NUT deformations of (X_λ, g_λ).

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2 Hyperkähler manifolds of type A_∞

In this section, we review the construction of hyperkähler manifolds of type A_∞ according to [10]. Although they can be constructed by Gibbons-Hawking ansatz [1], we need hyperkähler quotient construction in [10] for arguments in Section 3. Before the construction, we start with some basic definitions.

Definition 2.1. Let (X, g) be a Riemannian manifold of dimension $4n$ and I_1, I_2, I_3 be complex structures on X compatible with g. Then (X, g, I_1, I_2, I_3) is a hyperkähler manifold if (I_1, I_2, I_3) satisfies the quaternionic relations $I_1^2 = I_2^2 = I_3^2 = I_1 I_2 I_3 = -\mathrm{id}$ and each $\omega_i := g(I_i \cdot, \cdot)$ is closed for $i = 1, 2, 3$.

Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k = \mathbb{C} \oplus \mathbb{C}j$ be the quaternions and $\mathrm{Im}\mathbb{H} = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ the 3-dimensional subspace formed by the purely imaginary quaternions. We can combine three fundamental 2-forms into a single $\mathrm{Im}\mathbb{H}$-valued 2-form $\omega := \omega_1 i + \omega_2 j + \omega_3 k$. Then it turns out that three complex

structures and the metric are determined by the 2-form ω. Hence we call ω the hyperkähler structure on X instead of (g, I_1, I_2, I_3).
Let G be a Lie group acting on a manifold X. Then each element ξ of the Lie algebra of G generates a vector field $\xi^* \in \mathcal{X}(X)$ defined by $\xi^*_x := \frac{d}{dt}|_{t=0} x \exp(t\xi) \in T_xX$ for $x \in X$.

Definition 2.2. Suppose that the n-dimensional torus T^n with Lie algebra t^n acts smoothly on a hyperkähler manifold (X, ω) preserving the hyperkähler structure ω. Then the hyperkähler moment map $\mu : X \to \mathrm{Im}\mathbb{H} \otimes (t^n)^*$ for the action of T^n on X is defined by two properties (i) μ is T^n-invariant, (ii) $\langle d\mu_x(V), \xi\rangle = \omega(\xi^*_x, V_x) \in \mathrm{Im}\mathbb{H}$ for all $x \in X$ and $V \in T_xX$. Here $\langle , \rangle : \mathrm{Im}\mathbb{H} \otimes (t^n)^* \otimes t^n \to \mathrm{Im}\mathbb{H}$ is the natural pairing induced from the contraction.

Next we review the construction of hyperkähler manifolds of type A_∞. Let $S^{\mathbb{Z}}$ be the set of maps from the integers $\mathbb{Z}$ to a set S. Then each $x \in S^{\mathbb{Z}}$ is expressed in the sequence $x = (x_n)_{n\in\mathbb{Z}}$ where $x_n \in S$. Then we define a Hilbert space M in the infinite sequences of the quaternions by

$$M := \{v \in \mathbb{H}^{\mathbb{Z}};\ \|v\|^2_M < +\infty\},$$

where the Hilbert metric is given by

$$\langle u, v\rangle_M := \sum_{n\in\mathbb{Z}} u_n \bar{v}_n, \quad \|v\|^2_M := \langle v, v\rangle_M$$

for $u, v \in \mathbb{H}^{\mathbb{Z}}$.
Put $\mathbb{H}^{\mathbb{Z}}_0 := \{\Lambda \in \mathbb{H}^{\mathbb{Z}};\ \sum_{n\in\mathbb{Z}}(1 + |\Lambda_n|^2)^{-1} < +\infty\}$. Then for each $\Lambda \in \mathbb{H}^{\mathbb{Z}}_0$, we have the following Hilbert manifolds

$$\begin{aligned}
M_\Lambda &:= \Lambda + M = \{\Lambda + v;\ v \in M\},\\
U_\Lambda &:= \{g = (g_n)_{n\in\mathbb{Z}} \in (S^1)^{\mathbb{Z}};\ \|1_{\mathbb{Z}} - g\|^2_\Lambda < +\infty\},\\
\mathbf{u}_\Lambda &:= \{\xi = (\xi_n)_{n\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}};\ \|\xi\|^2_\Lambda < +\infty\},\\
G_\Lambda &:= \{g = (g_n)_{n\in\mathbb{Z}} \in U_\Lambda;\ \prod_{n\in\mathbb{Z}} g_n = 1\},\\
\mathbf{g}_\Lambda &:= \{\xi = (\xi_n)_{n\in\mathbb{Z}} \in \mathbf{u}_\Lambda;\ \sum_{n\in\mathbb{Z}} \xi_n = 0\},
\end{aligned}$$

where

$$\langle \xi, \eta\rangle_\Lambda := \sum_{n\in\mathbb{Z}} (1 + |\Lambda_n|^2)\xi_n \bar{\eta}_n, \quad \|\xi\|^2_\Lambda := \langle \xi, \xi\rangle_\Lambda$$

for $\xi, \eta \in \mathbb{C}^{\mathbb{Z}}$. Here, $1_{\mathbb{Z}} \in (S^1)^{\mathbb{Z}}$ is the constant map $(1_{\mathbb{Z}})_n := 1$. The convergence of $\prod_{n\in\mathbb{Z}} g_n$ and $\sum_{n\in\mathbb{Z}} \xi_n$ follows from the condition $\sum_{n\in\mathbb{Z}}(1 + \|\Lambda_n\|^2)^{-1} < +\infty$. Then G_Λ is a Hilbert Lie group whose Lie algebra is $\mathbf{g}_\Lambda$. We can define a right action of G_Λ on M_Λ by $xg := (x_n g_n)_{n\in\mathbb{Z}}$ for $x \in M_\Lambda$, $g \in G_\Lambda$. Here the product of x_n and g_n is given by regarding S^1 as the subset of $\mathbb{H}$ by the natural injections $S^1 \subset \mathbb{C} \subset \mathbb{H}$.

Since M is a left $\mathbb{H}$-module defined by $hv := (hv_n)_{n\in\mathbb{Z}}$ for $v \in M$ and $h \in \mathbb{H}$, M has a hyperkähler structure given by the left multiplication of i, j, k and inner product $\langle , \rangle_M$ on M. We denote by $I_1, I_2, I_3 \in \mathrm{End}(M)$ complex structures induced by the left multiplication by i, j, k, respectively. Thus M_Λ is an infinite dimensional hyperkähler manifold and the action of G_Λ on M_Λ preserves the hyperkähler structure. Then we define a map $\hat{\mu}_\Lambda : M_\Lambda \to \mathrm{Im}\mathbb{H} \otimes \mathbf{g}_\Lambda^*$ by

$$\langle \hat{\mu}_\Lambda(x), \xi \rangle := \sum_{n\in\mathbb{Z}} (x_n i \bar{x}_n - \Lambda_n i \bar{\Lambda}_n)\xi_n \ \in \mathrm{Im}\mathbb{H}$$

for $x \in M_\Lambda$, $\xi \in \mathbf{g}_\Lambda$. This $\hat{\mu}_\Lambda$ is the hyperkähler moment map for the action of infinite dimensional torus G_Λ on M_Λ.

Since $\hat{\mu}_\Lambda$ is G_Λ-invariant, then G_Λ acts on the inverse image $\hat{\mu}_\Lambda^{-1}(0)$. Hence we obtain the quotient space $\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ which is called hyperkähler quotient. In general, there is no guarantee that $\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ is a smooth manifold for every $\Lambda \in \mathbb{H}_0^{\mathbb{Z}}$. For the smoothness of $\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ we need the following condition.

Definition 2.3. An element $\Lambda \in \mathbb{H}_0^{\mathbb{Z}}$ is generic if $\Lambda_n i \bar{\Lambda}_n - \Lambda_m i \bar{\Lambda}_m \neq 0$ for all distinct $n, m \in \mathbb{Z}$.

Proposition 2.4. *For a generic $\Lambda \in \mathbb{H}_0^{\mathbb{Z}}$, the Lie group G_Λ acts freely on $\hat{\mu}_\Lambda^{-1}(0)$.*

Proof. Let $e_l \in \mathbf{g}_\Lambda$ be defined by $e_l := (e_{l,n})_{n\in\mathbb{Z}}$, where

$$e_{l,n} = \begin{cases} 1 & (n = l), \\ 0 & (n \neq l). \end{cases}$$

Since $\mathbf{g}_\Lambda$ is generated by elements $e_l - e_m$, then $x \in M_\Lambda$ satisfies $\hat{\mu}_\Lambda(x) = 0$ if and only if $\langle \hat{\mu}_\Lambda(x), e_l - e_m \rangle = 0$ for all $l, m \in \mathbb{Z}$. Hence the value of $x_n i \bar{x}_n - \Lambda_n i \bar{\Lambda}_n$ is independent of $n \in \mathbb{Z}$ for $x \in \hat{\mu}_\Lambda^{-1}(0)$.

Assume that there is a pair of $x \in \hat{\mu}_\Lambda^{-1}(0)$ and $g \in G_\Lambda$ satisfies $xg = x$. If $x_n \neq 0$ for any $n \in \mathbb{Z}$, then $g = 1$. Therefore we may assume $x_s = 0$ for some $s \in \mathbb{Z}$. Then we have $x_n i \bar{x}_n - \Lambda_n i \bar{\Lambda}_n = -\Lambda_s i \bar{\Lambda}_s$ for all $n \in \mathbb{Z}$, which implies

$$x_n i \bar{x}_n = \Lambda_n i \bar{\Lambda}_n - \Lambda_s i \bar{\Lambda}_s \neq 0$$

for $n \neq s$. Since $x_n = 0$ if and only if $x_n i \bar{x}_n = 0$, we have $x_n \neq 0$ if $n \neq s$. Thus we have shown $g_n = 1$ if $n \neq s$, and also g_s should be 1 from the condition $\prod_{n\in\mathbb{Z}} g_n = 1$. $\square$

Theorem 2.5 ([10])**.** *If $\Lambda \in \mathbb{H}_0^{\mathbb{Z}}$ is generic, then $\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ is a smooth manifold of dimension 4, and the hyperkähler structure on M_Λ induces a hyperkähler structure $\hat{\omega}_\Lambda$ on $\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$.*

Remark 2.1. In [10], the above theorem is proved in the case of

$$\Lambda_n = \begin{cases} ni & (n \geq 0), \\ nk & (n < 0). \end{cases}$$

But it is easy to show the theorem for the case of any generic $\Lambda \in \mathbb{H}_0^{\mathbb{Z}}$.

Take $\Lambda \in \mathbb{H}_0^{\mathbb{Z}}$ and $(e^{i\theta_n})_{n\in\mathbb{Z}} \in (S^1)^{\mathbb{Z}}$ and put $\Lambda' = (\Lambda_n e^{i\theta_n})_{n\in\mathbb{Z}}$. Then there is a canonical isomorphism of hyperkähler structure between $\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ and $\hat{\mu}_{\Lambda'}^{-1}(0)/G_{\Lambda'}$ as follows. Define a map $\hat{F} : \hat{\mu}_\Lambda^{-1}(0) \to \hat{\mu}_{\Lambda'}^{-1}(0)$ by

$$\hat{F}((x_n)_{n\in\mathbb{Z}}) := (x_n e^{i\theta_n})_{n\in\mathbb{Z}}.$$

Since $\hat{F}$ is equivariant with respect to G_Λ and $G_{\Lambda'}$-actions, we obtain $F : \hat{\mu}_\Lambda^{-1}(0)/G_\Lambda \to \hat{\mu}_{\Lambda'}^{-1}(0)/G_{\Lambda'}$, which preserves hyperkähler structures $\hat{\omega}_\Lambda$ and $\hat{\omega}_{\Lambda'}$. In this case we have $\Lambda_n i \bar{\Lambda}_n = \Lambda'_n i \bar{\Lambda}'_n$ for all $n \in \mathbb{Z}$. Converesely, if we take $\Lambda, \Lambda' \in \mathbb{H}_0^{\mathbb{Z}}$ to be $(\Lambda_n i \bar{\Lambda}_n)_{n\in\mathbb{Z}} = (\Lambda'_n i \bar{\Lambda}'_n)_{n\in\mathbb{Z}}$, then there is $(e^{i\theta_n})_{n\in\mathbb{Z}} \in (S^1)^{\mathbb{Z}}$ such that $\Lambda'_n = \Lambda_n e^{i\theta_n}$. Thus $\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ and $\hat{\mu}_{\Lambda'}^{-1}(0)/G_{\Lambda'}$ are isomorphic as hyperkähler manifolds if $(\Lambda_n i \bar{\Lambda}_n)_{n\in\mathbb{Z}} = (\Lambda'_n i \bar{\Lambda}'_n)_{n\in\mathbb{Z}}$. Therefore we may put $(X_\lambda, \omega_\lambda) := (\hat{\mu}_\Lambda^{-1}(0)/G_\Lambda, \hat{\omega}_\Lambda)$ for

$$\begin{aligned} \lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}} &:= \{(\Lambda_n i \bar{\Lambda}_n)_{n\in\mathbb{Z}} \in (\mathrm{Im}\mathbb{H})^{\mathbb{Z}};\ \Lambda \in \mathbb{H}_0^{\mathbb{Z}}\} \\ &= \{\lambda \in (\mathrm{Im}\mathbb{H})^{\mathbb{Z}};\ \sum_{n\in\mathbb{Z}} \frac{1}{1+|\lambda_n|} < +\infty\} \end{aligned}$$

and $\Lambda \in \hat{\nu}^{-1}(\lambda)$. Then the condition for X_λ to be smooth is written as follows.

Definition 2.6. An element $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$ is generic if $\lambda_n - \lambda_m \neq 0$ for all distinct $n, m \in \mathbb{Z}$.

Then Theorem 2.5 implies that $(X_\lambda, \omega_\lambda)$ is a smooth hyperkähler manifold for each generic $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$.
We denote by g_λ the hyperkähler metric induced from ω_λ.

Theorem 2.7 ([10])**.** *Let (X_λ, g_λ) be as above. Then the Riemannian metric g_λ is complete.*

We denote by $[x] \in \hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ the equivalence class represented by $x \in \hat{\mu}_\Lambda^{-1}(0)$. Then by putting

$$[x]g := [(\cdots, x_{-1}, x_0 g, x_1, \cdots, x_n, \cdots)]$$

for $x = (x_n)_{n\in\mathbb{Z}} = (\cdots, x_{-1}, x_0, x_1, \cdots, x_n, \cdots) \in \hat{\mu}_\Lambda^{-1}(0)$ and $g \in S^1$, we have the action of S^1 on X_λ. Then the hyperkähler moment map $\mu_\lambda : X_\lambda \to \mathrm{Im}\mathbb{H} = \mathbb{R}^3$ is given by

$$\mu_\lambda([x]) := x_0 i \bar{x}_0 - \lambda_0 \in \mathrm{Im}\mathbb{H}.$$

Since $x \in \hat{\mu}_\Lambda^{-1}(0)$, we have $x_0 i \bar{x}_0 - \lambda_0 = x_n i \bar{x}_n - \lambda_n$, where we identify $(t^1)^*$ with $\mathbb{R}$ by taking a generator of t^1.
If we put $Stab([x]) := \{g \in S^1;\ [x]g = [x]\}$ for $[x] \in X_\lambda$, then it is obvious that $Stab([x]) = \{1\}$ if and only if $x_n \neq 0$ for any $n \in \mathbb{Z}$, otherwise $Stab([x]) = S^1$. Hence we have a principal S^1-bundle $\mu_\lambda|_{X_\lambda^*} : X_\lambda^* \to Y_\lambda$ where

$$\begin{aligned} X_\lambda^* &:= \{[x] \in X_\lambda;\ x_n \neq 0 \text{ for all } n \in \mathbb{Z}\}, \\ Y_\lambda &:= \mathrm{Im}\mathbb{H}\backslash\{-\lambda_n;\ n \in \mathbb{Z}\}. \end{aligned}$$

Definition 2.8. An S^1-action on a 4-dimensional hyperkähler manifold (X, ω) is tri-Hamiltonian if the action preserves ω and there exists a hyperkähler moment map $\mu : X \to \mathrm{Im}\mathbb{H}$ for the action of S^1.

Theorem 2.9 ([9])**.** *There exists a canonical one-to-one correspondence between the followings; (i) a hyperkähler manifold of real dimension 4 with free tri-Hamiltonian S^1-action, (ii) a principal S^1-bundle $\mu : X \to Y$ where Y is an open subset of $\mathbb{R}^3$, and an S^1-connection A on X and a positive valued harmonic function Φ on Y such that $\frac{dA}{2\sqrt{-1}} = \mu^*(*d\Phi)$. Here $*$ is the Hodge star operator with respect to the Euclidean metric on $\mathbb{R}^3$.*

Here we see a sketch of the proof, and the details can be seen in [10].
Let (X, ω) be a hyperkähler manifold of dimension 4 with a free S^1-action preserving ω, and $\mu : X \to \mathrm{Im}\mathbb{H}$ the hyperkähler moment map. Then (Y, Φ, A) is given by the followings. Let Y be the image $\mu(X)$. Then the function $\Phi : Y \to \mathbb{R}$ is defined by

$$\frac{1}{\Phi(\mu(x))} := 4g_x(\xi_x^*, \xi_x^*)$$

for $x \in X$, where g is the hyperkähler metric and $\xi := \sqrt{-1}$ is a generator of Lie algebre of the Lie group S^1. For each $x \in X$, we denote by $V_x \subset T_x X$ the subspace spanned by ξ_x^*. Then the S^1-connection A on X is defined

by the horizontal distribution $(H_x)_{x\in X}$ where $H_x \subset T_xX$ is the orthogonal complement of V_x.
Conversely, let $\mu = (\mu_1, \mu_2, \mu_3) : X \to Y \subset \mathrm{Im}\mathbb{H}$ be a principal S^1-bundle and (Φ, A) a pair consisting a positive valued harmonic function and a connection of the S^1-bundle with $\frac{dA}{2\sqrt{-1}} = \mu^*(*d\Phi)$. Then the hyperkähler structure $\omega = (\omega_1, \omega_2, \omega_3) \in \Omega^2(X) \otimes \mathrm{Im}\mathbb{H}$ is defined by

$$\omega_1 := d\mu_1 \wedge \frac{A}{2\sqrt{-1}} + \mu^*\Phi d\mu_2 \wedge d\mu_3,$$
$$\omega_2 := d\mu_2 \wedge \frac{A}{2\sqrt{-1}} + \mu^*\Phi d\mu_3 \wedge d\mu_1,$$
$$\omega_3 := d\mu_3 \wedge \frac{A}{2\sqrt{-1}} + \mu^*\Phi d\mu_1 \wedge d\mu_2.$$

Then it follows from the condition $\frac{dA}{2\sqrt{-1}} = \mu^*(*d\Phi)$ that ω is closed.
Theorem 2.9 gives the positive valued harmonic function Φ_λ on Y_λ and S^1-connection A_λ on X_λ^*. It is known in [10] that Φ_λ is given by

$$\Phi_\lambda(\zeta) = \frac{1}{4}\sum_{n\in\mathbb{Z}} \frac{1}{|\zeta + \lambda_n|}$$

for $\zeta \in Y_\lambda$.
Let (X, ω) be a hyperkähler manifold satisfying the condition (i) of Theorem 2.9, and (Y, Φ, A) be what corresponds to (X, ω) satisfying the condition (ii). Denote by g the hyperkähler metric of ω and let $\mathrm{vol}_g(B)$ be the volume of a subset $B \subset X$.

Lemma 2.10. *Let $U \subset \mathrm{Im}\mathbb{H}$ be an open set. Then we have the following formula*

$$\mathrm{vol}_g(\mu^{-1}(U)) = \pi \int_{\zeta\in U} \Phi(\zeta) d\zeta_1 d\zeta_2 d\zeta_3,$$

where $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathrm{Im}\mathbb{H}$ is the Cartesian coordinate.

Proof. It suffices to show the assertion for all open set $U \subset Y$.
First of all we suppose that the principal S^1-bundle $\mu : \mu^{-1}(U) \to U$ is trivial. Then we can take a C^∞ trivialization $\sigma : U \to \mu^{-1}(U)$ and define C^∞ map $t : \mu^{-1}(U) \to \mathbb{R}/2\pi\mathbb{Z}$ by $t(\sigma(\zeta)e^{i\theta}) := \theta$ for $\zeta \in U$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and obtain a local coordinate (t, μ_1, μ_2, μ_3) on $\mu^{-1}(U)$. Since we may write $dt = -\sqrt{-1}A + \sum_{l=1}^3 a_l d\mu^l$ for some $a_1, a_2, a_3 \in C^\infty(\mu^{-1}(U))$, the volume

form vol_g is given by

$$\begin{aligned}\mathrm{vol}_g &= \frac{\mu^*\Phi}{2}(-\sqrt{-1})d\mu_1\wedge d\mu_2\wedge d\mu_3\wedge A\\ &= \frac{\mu^*\Phi}{2}d\mu_1\wedge d\mu_2\wedge d\mu_3\wedge dt.\end{aligned}$$

Thus we have

$$\begin{aligned}\mathrm{vol}_g(\mu^{-1}(U)) &= \int_{\mu^{-1}(U)}\mathrm{vol}_g\\ &= \int_{\mu^{-1}(U)}\frac{\mu^*\Phi}{2}d\mu_1\wedge d\mu_2\wedge d\mu_3\wedge dt\\ &= \pi\int_{\zeta\in U}\Phi(\zeta)d\zeta_1 d\zeta_2 d\zeta_3.\end{aligned}$$

For a general $U\subset Y$, we take open sets $U_1, U_2, \cdots, U_m\subset Y$ such that each principal S^1-bundle $\mu^{-1}(U_\alpha)\to U_\alpha$ is trivial and $\coprod_{\alpha=1}^m U_\alpha\subset U\subset \overline{\coprod_{\alpha=1}^m U_\alpha}$. Then we have

$$\begin{aligned}\mathrm{vol}_g(\mu^{-1}(U)) &= \sum_{\alpha=1}^m \mathrm{vol}_g(\mu^{-1}(U_\alpha))\\ &= \sum_{\alpha=1}^m \pi\int_{\zeta\in U_\alpha}\Phi(\zeta)d\zeta_1 d\zeta_2 d\zeta_3\\ &= \pi\int_{\zeta\in U}\Phi(\zeta)d\zeta_1 d\zeta_2 d\zeta_3.\end{aligned}$$

□

3 The upper bound for the volume growth

For a Riemannian manifold (X,g) and a point $p_0\in X$, we denote by $V_g(p_0,r)$ the volume of a ball $B_g(p_0,r):=\{p\in X;\ d_g(p_0,p)<r\}$ with respect to the Riemannian distance d_g. In this section, we will evaluate the upper bound for $V_{g_\lambda}(p_0,r)$ for the hyperkähler manifold (X_λ, g_λ).
In Section 2, we supposed $\lambda\in(\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$ to be generic for the smoothness of (X_λ, g_λ). But the function $V_{g_\lambda}(p_0,r)$ is determined only by the Riemannian measure vol_{g_λ} and the Riemannian distance d_{g_λ}. Even if $\lambda\in(\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$ is taken not to be generic, vol_{g_λ} and d_{g_λ} can be extended to X_λ naturally. Hence we take $\lambda\in(\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$ not necessary to be generic from now on.

Fix $p_0 \in X_\lambda$ to be $\mu_\lambda(p_0) = -\lambda_0$. We may assume $\lambda_0 = 0$ since the hyperkähler quotient constructed from $(\lambda_n - \lambda_0)_{n\in\mathbb{Z}}$ is isometric to (X_λ, g_λ). For each $R > 0$, put

$$\varphi_\lambda(R) := \sum_{n\in\mathbb{Z}} \frac{R}{R + |\lambda_n|},$$

and $\varphi_\lambda(0) := \lim_{R\to+0} \varphi_\lambda(R)$.

Proposition 3.1. *We have the following inequality*

$$d_{g_\lambda}(p_0, p)^2 \geq Q_- |\mu_\lambda(p)| \cdot \varphi_\lambda(|\mu_\lambda(p)|)$$

for any $p \in X_\lambda$, where $Q_- = 1/8$.

Proof. Take $\Lambda = (\Lambda_n)_{n\in\mathbb{Z}} \in \mathbb{H}_0^{\mathbb{Z}}$ to be $\Lambda_n i \bar{\Lambda}_n = \lambda_n$ for all $n \in \mathbb{Z}$. We fix $x \in \hat{\mu}_\Lambda^{-1}(0)$ such that $[x] = p$. If we regard $\hat{\mu}_\Lambda^{-1}(0)$ as the infinite dimensional Riemannian submanifold of M_Λ, then the quotient map $\hat{\mu}_\Lambda^{-1}(0) \to X_\lambda = \hat{\mu}_\Lambda^{-1}(0)/G_\Lambda$ is a Riemannian submersion and G_Λ acts on $\hat{\mu}_\Lambda^{-1}(0)$ as an isometry. Then the horizontal lift of the geodesic from p_0 to p has the same length as $d_{g_\lambda}(p_0, p)$. Since the Riemannian distance between Λ and $x' \in \hat{\mu}_\Lambda^{-1}(0)$ is larger than $\|\Lambda - x'\|_M$, then we have

$$d_{g_\lambda}(p_0, p) \geq \inf_{\sigma\in G_\Lambda} \|\Lambda - x\sigma\|_M.$$

If we put $\Lambda = (\alpha_n + \beta_n j)_{n\in\mathbb{Z}}$, $x = (z_n + w_n j)_{n\in\mathbb{Z}}$ and $\sigma = (e^{i\theta_n})_{n\in\mathbb{Z}}$, then

$$\|\Lambda - x\sigma\|_M^2 = \sum_{n\in\mathbb{Z}} (|\alpha_n - z_n e^{i\theta_n}|^2 + |\beta_n - w_n e^{-i\theta_n}|^2).$$

If the function $f_n(t) := |\alpha_n - z_n e^{it}|^2 + |\beta_n - w_n e^{-it}|^2$ attains its minimum at θ_n for each $n \in \mathbb{Z}$, then $\sigma = (e^{i\theta_n})_{n\in\mathbb{Z}}$ satisfies$\|\Lambda - x\sigma\|_M \leq \inf_{\tau\in G_\Lambda} \|\Lambda - x\tau\|_M$. From the equations

$$\lambda_n = \Lambda_n i \overline{\Lambda}_n = i(|\alpha_n|^2 - |\beta_n|^2) - 2\alpha_n\beta_n k,$$
$$\zeta + \lambda_n = x_n i \overline{x}_n = i(|z_n|^2 - |w_n|^2) - 2z_n w_n k,$$

we have

$$f_n(t) = |\lambda_n| + |\zeta + \lambda_n| - 2Re\{(\alpha_n \bar{z}_n + \bar{\beta}_n w_n)e^{-it}\},$$
$$|\alpha_n \bar{z}_n + \bar{\beta}_n w_n|^2 = \frac{1}{2}(|\lambda_n||\zeta + \lambda_n| + \langle \lambda_n, \zeta + \lambda_n \rangle_{\mathbb{R}^3}),$$

where $\langle , \rangle_{\mathbb{R}^3}$ is the standard inner product on $\mathrm{Im}\mathbb{H} = \mathbb{R}^3$.
Suppose $n \neq 0$. If $\alpha_n \bar{z}_n + \bar{\beta}_n w_n \neq 0$, then we put $e^{i\theta_n} := \frac{\alpha_n \bar{z}_n + \bar{\beta}_n w_n}{|\alpha_n \bar{z}_n + \bar{\beta}_n w_n|}$. If

$\alpha_n \bar{z}_n + \bar{\beta}_n w_n = 0$, we may put $\theta_n := 0$ since $f_n(t)$ is constant.
Let $S := \{n \in \mathbb{Z};\ \alpha_n \bar{z}_n + \bar{\beta}_n w_n = 0\}$. For each $n \notin S$, we have

$$\begin{aligned} |\alpha_n - z_n e^{i\theta_n}|^2 + |\beta_n - w_n e^{-i\theta_n}|^2 &= \frac{|\zeta|^2}{|\lambda_n| + |\zeta + \lambda_n| + 2|\alpha_n \bar{z}_n + \bar{\beta}_n w_n|} \\ &\leq \frac{|\zeta|^2}{|\lambda_n|}. \end{aligned}$$

Hence we deduce

$$\sum_{n\in\mathbb{Z}} |\Lambda_n - x_n e^{i\theta_n}|^2 \leq \sum_{n\in S} |\Lambda_n - x_n|^2 + \sum_{n\notin S} \frac{|\zeta|^2}{|\lambda_n|} < +\infty.$$

Thus we obtain $\sum_{n\in\mathbb{Z}\setminus\{0\}} |\Lambda_n|^2 |1 - e^{i\theta_n}|^2 < +\infty$, which ensures the convergence of $\Pi_{n\in\mathbb{Z}\setminus\{0\}} e^{i\theta_n}$.
It follows from $\lambda_0 = 0$ that $\alpha_0 = \beta_0 = 0$. Then $f_0(t)$ is constant. Since $f_0(t)$ attains its minimum at any θ_0, then we may put $e^{i\theta_0} := \Pi_{n\in\mathbb{Z}\setminus\{0\}} e^{-i\theta_n}$. Thus the function $\|\Lambda - x\tau\|_M$ attains its minimum at $\tau = \sigma = (e^{i\theta_n})_{n\in\mathbb{Z}} \in G_\Lambda$. For this σ, we have

$$\begin{aligned} \|\Lambda - x\sigma\|_M^2 &= \sum_{n\in\mathbb{Z}} (|\lambda_n| + |\zeta + \lambda_n| - \sqrt{2}\sqrt{|\lambda_n||\zeta + \lambda_n| + \langle \lambda_n, \zeta + \lambda_n \rangle_{\mathbb{R}^3}}) \\ &= \sum_{n\in\mathbb{Z}} \frac{|\zeta|^2}{|\lambda_n| + |\zeta + \lambda_n| + \sqrt{2}\sqrt{|\lambda_n||\zeta + \lambda_n| + \langle \lambda_n, \zeta + \lambda_n \rangle_{\mathbb{R}^3}}} \\ &\geq \sum_{n\in\mathbb{Z}} \frac{|\zeta|^2}{8|\lambda_n| + 2|\zeta|} \geq \frac{1}{8}|\zeta| \cdot \varphi_\lambda(|\zeta|). \end{aligned}$$

Then the assertion is obtained by putting $Q_- = \frac{1}{8}$. $\square$

Put $B_R := \{\zeta \in \mathrm{Im}\mathbb{H};\ |\zeta| < R\}$. Next we discuss the upper bound for the volume of $\mu_\lambda^{-1}(B_R)$.

Lemma 3.2. *We define two functions $N_\lambda(R)$ and $\psi_\lambda(R)$ by*

$$\begin{aligned} N_\lambda(R) &:= \{n \in \mathbb{Z};\ |\lambda_n| \leq R\}, \\ \psi_\lambda(R) &:= \sharp N_\lambda(R) + \sum_{n\notin N_\lambda(R)} \frac{1}{|\lambda_n|} R \end{aligned}$$

for $R \geq 0$. Then we have

$$\varphi_\lambda(R) \leq \psi_\lambda(R) \leq 2\varphi_\lambda(R).$$

Proof. We define two functions $p(x)$ and $q(x)$ by

$$p(x) := \frac{x}{1+x}, \quad q(x) := \min\{1, x\}$$

for $x \geq 0$. Then the inequalities

$$p(x) \leq q(x) \leq 2p(x)$$

hold for each $x \geq 0$. Therefore the assertion follows from

$$\varphi_\lambda(R) = \sum_{n\in\mathbb{Z}} p\Big(\frac{R}{|\lambda_n|}\Big), \quad \psi_\lambda(R) = \sum_{n\in\mathbb{Z}} q\Big(\frac{R}{|\lambda_n|}\Big).$$

□

Lemma 3.3. *For $\alpha \geq 1$ and $R \geq 0$, we have $\varphi_\lambda(\alpha R) \leq \alpha\varphi_\lambda(R)$ and $\psi_\lambda(\alpha R) \leq \alpha\psi_\lambda(R)$.*

Proof. Since $\frac{\varphi_\lambda(R)}{R}$ is strictly decreasing for R, we have

$$\varphi_\lambda(\alpha R) = \frac{\varphi_\lambda(\alpha R)}{\alpha R}\alpha R \leq \frac{\varphi_\lambda(R)}{R}\alpha R = \alpha\varphi_\lambda(R)$$

for $\alpha \geq 1$ and $R \geq 0$. It follows from the same argument that $\psi_\lambda(\alpha R) \leq \alpha\psi_\lambda(R)$. □

Proposition 3.4. *There is a constant $P_+ > 0$ independent of λ and R such that*

$$\mathrm{vol}_{g_\lambda}(\mu_\lambda^{-1}(B_R)) \leq P_+ R^2 \varphi_\lambda(R)$$

for all $R \geq 0$.

Proof. From Lemma 2.10, the volume of $\mu_\lambda^{-1}(B_R)$ is given by

$$\begin{aligned}
\mathrm{vol}_{g_\lambda}(\mu_\lambda^{-1}(B_R)) &= \pi \int_{\zeta\in B_R} \Phi_\lambda(\zeta) d\zeta_1 d\zeta_2 d\zeta_3 \\
&= \frac{\pi}{4} \sum_{n\in\mathbb{Z}} \int_{\zeta\in B_R} \frac{1}{|\zeta + \lambda_n|} d\zeta_1 d\zeta_2 d\zeta_3.
\end{aligned}$$

Let $(r \geq 0, \Theta)$ be the polar coordinate over $\mathrm{Im}\mathbb{H} = \mathbb{R}^3$, where Θ is a coordinate on $S^2 = \partial B_1$. If $n \in N_\lambda(R)$, then the change of variables $\zeta' = \zeta + \lambda_n$ gives

$$\begin{aligned}
\int_{\zeta\in B_R} \frac{1}{|\zeta + \lambda_n|} d\zeta_1 d\zeta_2 d\zeta_3 &\leq \int_{\zeta'\in B_{R+|\lambda_n|}} \frac{1}{|\zeta'|} d\zeta'_1 d\zeta'_2 d\zeta'_3 \\
&= 4\pi \int_0^{R+|\lambda_n|} r dr \\
&= 2\pi(R + |\lambda_n|)^2 \leq 8\pi R^2.
\end{aligned}$$

If $n \notin N_\lambda(R)$, the mean value property of harmonic functions gives

$$\int_{\zeta \in B_R} \frac{1}{|\zeta + \lambda_n|} d\zeta_1 d\zeta_2 d\zeta_3 \quad = \quad \frac{4\pi R^3}{3} \frac{1}{|\lambda_n|},$$

since $\frac{1}{|\zeta+\lambda_n|}$ is harmonic on B_R. Hence the upper bound for $\mathrm{vol}_{g_\lambda}(\mu_\lambda^{-1}(B_R))$ is given by

$$\begin{aligned} \mathrm{vol}_{g_\lambda}(\mu_\lambda^{-1}(B_R)) &\leq 2\pi^2 \sharp N_\lambda(R) \cdot R^2 + \frac{\pi^2}{3} \sum_{n \notin N_\lambda(R)} \frac{R}{|\lambda_n|} \cdot R^2 \\ &\leq 2\pi^2 \psi_\lambda(R) R^2 \leq 4\pi^2 \varphi_\lambda(R) R^2. \end{aligned}$$

Then we have the assertion by putting $P_+ := 4\pi^2$. $\square$

Let $\theta_{\lambda,C}(R) := C\varphi_\lambda(R)R^2$, $\tau_{\lambda,C}(R) := C\varphi_\lambda(R)R$ for $C > 0$, and $\tau_{\lambda,C}^{-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be the inverse function of $\tau_{\lambda,C}$.

Proposition 3.5. *Let P_+ and Q_- be as in Proposition 3.1 and 3.4. Then the inequality*

$$V_{g_\lambda}(p_0, r) \leq \theta_{\lambda,P_+} \circ \tau_{\lambda,Q_-}^{-1}(r^2)$$

holds for all $r \geq 0$.

Proof. Since $p \in B_{g_\lambda}(p_0, \sqrt{\tau_{\lambda,Q_-}(R)})$ satisfies $d_{g_\lambda}(p_0, p)^2 \leq \tau_{\lambda,Q_-}(R)$, then from Proposition 3.1 we obtain

$$\tau_{\lambda,Q_-}(|\mu_\lambda(p)|) = Q_-|\mu_\lambda(p)|\varphi(|\mu_\lambda(p)|) \leq \tau_{\lambda,Q_-}(R).$$

Since the function $\tau_{\lambda,Q_-}(R)$ is strictly increasing, we have $|\mu_\lambda(p)| \leq R$. Then we have $B_{g_\lambda}(p_0, \sqrt{\tau_{\lambda,Q_-}(R)}) \subset \mu_\lambda^{-1}(B_R)$. Then it follows from Proposition 3.4 that we have

$$\mathrm{vol}_{g_\lambda}(B_{g_\lambda}(p_0, \sqrt{\tau_{\lambda,Q_-}(R)})) \leq \mathrm{vol}_{g_\lambda}(\mu_\lambda^{-1}(B_R)) \leq P_+ R^2 \varphi(R) = \theta_{\lambda,P_+}(R).$$

Set $r = \sqrt{\tau_{\lambda,Q_-}(R)}$. Then substituting for R the inverse function $\tau_{\lambda,Q_-}^{-1}(r^2)$, we obtain the result. $\square$

4 The lower bound for the volume growth

In the previous section we obtained the upper bound for $V_{g_\lambda}(p_0, r)$. The purpose of this section is to obtain the inequality

$$\theta_{\lambda,P_-} \circ \tau_{\lambda,Q_+}^{-1}(r^2) \leq V_{g_\lambda}(p_0, r),$$

where $P_-, Q_+ > 0$ are constants independent of λ and r. In similar consideration as in Section 3, it seems that the following two inequalities should be shown,

$$\begin{aligned} \mathrm{vol}_{g_\lambda}(\mu_\lambda^{-1}(B_R)) &\geq P_- R^2 \varphi_\lambda(R), &(1)\\ d_{g_\lambda}(p_0, p)^2 &\leq Q_+ |\mu_\lambda(p)| \cdot \varphi_\lambda(|\mu_\lambda(p)|). &(2)\end{aligned}$$

But it seems to be hard to show the inequality (2) for the author. Accordingly we shall show two modified inequalities; one is a stronger inequality than (1), and the other is weaker than (2). First of all we consider the former estimate. Let $U \subset S^2$ be a measurable set and put

$$B_{R,U} := \{t\zeta \in \mathrm{Im}\mathbb{H};\ 0 \leq t \leq R,\ \zeta \in U\}.$$

We denote by m_{S^2} the measure induced from the Riemannian metric with constant curvature over S^2 whose total measure is given by $m_{S^2}(S^2) = 4\pi$. First of all we consider the lower bound for $\mathrm{vol}_{g_\lambda}(\mu_\lambda^{-1}(B_{R,U}))$.

Proposition 4.1. *There is a constant $C_- > 0$ independent of λ, R and $U \subset S^2$, which satisfies*

$$\mathrm{vol}_{g_\lambda}(\mu_\lambda^{-1}(B_{R,U})) \geq C_- m_{S^2}(U) R^2 \cdot \varphi_\lambda(R).$$

Proof. From the triangle inequality, we have

$$\begin{aligned} \mathrm{vol}_{g_\lambda}(\mu_\lambda^{-1}(B_{R,U})) &\geq \frac{\pi}{4}\sum_{n\in\mathbb{Z}} \int_{\zeta\in B_{R,U}} \frac{1}{|\zeta| + |\lambda_n|} d\zeta_1 d\zeta_2 d\zeta_3 \\ &= \frac{\pi}{4}\sum_{n\in\mathbb{Z}} \int_{r\in[0,R]} \int_{\Theta\in U} \frac{1}{r + |\lambda_n|} r^2 dr dm_{S^2} \\ &= \frac{\pi m_{S^2}(U)}{4} \sum_{n\in\mathbb{Z}} \int_{r\in[0,R]} \frac{1}{r + |\lambda_n|} r^2 dr. \end{aligned}$$

If $n \in N_\lambda(R)$, then

$$\int_{r\in[0,R]} \frac{1}{r + |\lambda_n|} r^2 dr \geq \int_{r\in[0,R]} \frac{1}{r + R} r^2 dr = (\log 2 - \frac{1}{2}) R^2.$$

If $n \notin N_\lambda(R)$, then

$$\begin{aligned} \int_{r\in[0,R]} \frac{1}{r + |\lambda_n|} r^2 dr &= \int_{r\in[0,R]} \frac{1}{r + |\lambda_n|} r^2 dr \\ &= |\lambda_n|^2 \Big\{ -\frac{R}{|\lambda_n|} + \frac{1}{2}\Big(\frac{R}{|\lambda_n|}\Big)^2 + \log\Big(1 + \frac{R}{|\lambda_n|}\Big)\Big\}. \end{aligned}$$

Since the inequality

$$\log(1+x) \geq x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$$

holds for $x \geq 0$, then we have

$$\begin{aligned}\int_{r\in[0,R]} \frac{1}{r+|\lambda_n|} r^2 dr &\geq |\lambda_n|^2 \Big\{ \frac{1}{3}\Big(\frac{R}{|\lambda_n|}\Big)^3 - \frac{1}{4}\Big(\frac{R}{|\lambda_n|}\Big)^4 \Big\} \\ &\geq \frac{1}{12}\frac{R^3}{|\lambda_n|}.\end{aligned}$$

By taking $\frac{4C_-}{\pi} := \min\{\log 2 - \frac{1}{2}, \frac{1}{12}\} > 0$, we have

$$\mathrm{vol}_{g_\lambda}(\mu_\lambda^{-1}(B_{R,U})) \geq C_- m_{S^2}(U) R^2 \cdot \psi_\lambda(R) \geq C_- m_{S^2}(U) R^2 \cdot \varphi_\lambda(R).$$

$\square$

Next we need the upper bound for $d_{g_\lambda}(p_0,p)$. If we can calculate the length of a piecewise smooth path from p_0 to p, then the length is larger than $d_{g_\lambda}(p_0,p)$. Here we take the path as follows.
Put $\zeta = \mu_\lambda(p)$. For $|\zeta| \leq 1$, we define γ_p to be the geodesic from p_0 to p. If $|\zeta| > 1$, we define a function $\delta : [1,|\zeta|] \to \mathrm{Im}\mathbb{H}$ by $\delta(t) := t\frac{\zeta}{|\zeta|}$ and take the horizontal lift $\tilde{\delta} : [1,|\zeta|] \to X_\lambda$ of δ with respect to the connection A_λ such that $\tilde{\delta}(|\zeta|) = p$. If there exists a point $t \in [1,|\zeta|]$ such that $\delta(t) = -\lambda_n$ for some $n \in \mathbb{Z}$, then $\tilde{\delta}$ is not always smooth but continuous and piecewise smooth. Then we obtain a path γ_p by connecting the geodesic from p_0 to $\tilde{\delta}(1)$ and $\tilde{\delta}$.
The length of γ_p is given by $d_{g_\lambda}(p_0,\tilde{\delta}(1)) + l_\lambda(\mu_\lambda(p))$ where $l_\lambda : \mathrm{Im}\mathbb{H} \to \mathbb{R}$ is defined by

$$l_\lambda(\zeta) := \int_1^{|\zeta|} \sqrt{\Phi_\lambda\Big(t\frac{\zeta}{|\zeta|}\Big)}dt$$

for $|\zeta| \geq 1$, and $l_\lambda(\zeta) := 0$ for $|\zeta| \leq 1$.
If there exists a point $t \in [1,|\zeta|]$ such that $\delta(t) = -\lambda_n$, then $\sqrt{\Phi_\lambda\Big(t\frac{\zeta}{|\zeta|}\Big)}$ is not continuous at $t = |\lambda_n|$. But the integral $\int_1^{|\zeta|}\sqrt{\Phi_\lambda\Big(t\frac{\zeta}{|\zeta|}\Big)}dt$ is well-defined since the integral

$$\int_{b-\varepsilon}^{b+\varepsilon} \sqrt{\frac{1}{a|t-b|} + h(t)dt}$$

is a finite value for any constant $a > 0$, $b \in \mathbb{R}$ and smooth function $h(t)$.
Now we have $d_{g_\lambda}(p_0, p) \leq L_\lambda + l_\lambda(\mu_\lambda(p))$, where

$$L_\lambda := \sup_{p \in \mu_\lambda^{-1}(\bar{B}_1)} d_{g_\lambda}(p_0, p) < +\infty.$$

Proposition 4.2. *There is a constant $C_+ > 0$ independent of λ and R which satisfies*

$$\int_{\Theta \in \partial B_1} l_\lambda(R\Theta) dm_{S^2} \leq 4\pi \sqrt{C_+ R \varphi_\lambda(R)}.$$

Proof. We may suppose $R \geq 1$, since the left hand side of the assertion is equal to 0 if $R < 1$. The definition of l_λ gives

$$\begin{aligned} \int_{\Theta \in \partial B_1} l_\lambda(R\Theta) dm_{S^2} &\leq \int_{\Theta \in \partial B_1} \int_1^R \sqrt{\Phi_\lambda(t\Theta)} dt dm_{S^2} \\ &= \int_{\zeta \in B_R \setminus B_1} \frac{\sqrt{\Phi_\lambda(\zeta)}}{|\zeta|^2} d\zeta, \end{aligned} \tag{3}$$

where $d\zeta = d\zeta_1 d\zeta_2 d\zeta_3$.
Take $m \in \mathbb{Z}_{\geq 0}$ to be $2^m \leq R < 2^{m+1}$. Then the Cauchy-Schwarz inequality gives

$$\begin{aligned} (3) &= \sum_{l=0}^{m-1} \int_{\zeta \in B_{2^{l+1}} \setminus B_{2^l}} \frac{\sqrt{\Phi_\lambda(\zeta)}}{|\zeta|^2} d\zeta + \int_{\zeta \in B_R \setminus B_{2^m}} \frac{\sqrt{\Phi_\lambda(\zeta)}}{|\zeta|^2} d\zeta \\ &\leq \sum_{l=0}^{m-1} \sqrt{\int_{(r,\Theta) \in B_{2^{l+1}} \setminus B_{2^l}} \frac{r^2 dr d\Theta}{r^4}} \sqrt{\int_{\zeta \in B_{2^{l+1}} \setminus B_{2^l}} \Phi_\lambda(\zeta) d\zeta} \\ &\quad + \sqrt{\int_{(r,\Theta) \in B_R \setminus B_{2^m}} \frac{r^2 dr d\Theta}{r^4}} \sqrt{\int_{\zeta \in B_R \setminus B_{2^m}} \Phi_\lambda(\zeta) d\zeta}. \end{aligned}$$

From Proposition 3.4, the inequalities

$$\int_{\zeta \in B_t \setminus B_{t'}} \Phi_\lambda(\zeta) d\zeta \leq \int_{\zeta \in B_t} \Phi_\lambda(\zeta) d\zeta \leq \frac{P_+}{\pi} t^2 \varphi_\lambda(t)$$

hold for any $0 \leq t' \leq t$. Therefore the assertion follows from

$$(3) \leq \sqrt{4\pi} \sum_{l=0}^{m-1} \sqrt{\frac{1}{2^l} - \frac{1}{2^{l+1}}} \sqrt{\frac{P_+}{\pi} 2^{2(l+1)} \varphi_\lambda(2^{l+1})}$$

$$
\begin{aligned}
& +\sqrt{4\pi}\sqrt{\frac{1}{2^m}-\frac{1}{R}}\sqrt{\frac{P_+}{\pi}R^2\varphi_\lambda(R)} \\
\leq\ & 2\sum_{l=0}^{m-1}\sqrt{2}^{l+1}\sqrt{P_+\varphi_\lambda(R)}+2\sqrt{P_+R\varphi_\lambda(R)} \\
\leq\ & 2(3+\sqrt{2})\sqrt{P_+R\varphi_\lambda(R)}
\end{aligned}
$$

by putting $C_+ = (\frac{3+\sqrt{2}}{2\pi})^2 P_+$. ☐

Since $R\varphi_\lambda(R)$ diverges to $+\infty$ for $R \to +\infty$, there is a constant $R_0 > 0$ which satisfies

$$
4\pi L_\lambda + \int_{\Theta\in S^2} l_\lambda(R\Theta)dm_{S^2} \leq 4\pi\sqrt{2C_+R\varphi_\lambda(R)} \tag{4}
$$

for all $R \geq R_0$. Now we put

$$
U_{R,T} := \{\Theta \in S^2;\ L_\lambda + l_\lambda(R\Theta) \leq \sqrt{TR\varphi_\lambda(R)}\}
$$

for $R, T > 0$. For a fixed $R > 0$, we introduce a function $F(\Theta)$ on S^2 by

$$
F(\Theta) := L_\lambda + l_\lambda(R\Theta).
$$

Then $U_{R,T}$ is the inverse image $F^{-1}([0, \sqrt{TR\varphi_\lambda(R)}])$, hence $U_{R,T}$ is a measurable set.

Lemma 4.3. *There exists a sufficiently large $R_0 > 0$ and we have*

$$
m_{S^2}(U_{R,T}) \geq 4\pi\frac{\sqrt{T}-\sqrt{2C_+}}{\sqrt{T}-\sqrt{Q_-}}
$$

for $R \geq R_0$ and $T > 2C_+$.

Proof. From (4), we have the upper bound

$$
4\pi\sqrt{2C_+R\varphi_\lambda(R)} \geq \int_{\Theta\in S^2} F(\Theta)dm_{S^2}. \tag{5}
$$

Since $F(\Theta) := L_\lambda + l_\lambda(R\Theta) \geq d_{g_\lambda}(p_0, p)$ and $\mu_\lambda(p) = \zeta = R\Theta$, Lemma 3.1 yields the lower bound for $F(\Theta)$,

$$
F(\Theta) \geq \sqrt{Q_-R\varphi_\lambda(R)}.
$$

On the complement $U_{R,T}^c$ of $U_{R,T}$ in S^2, we have $F(\Theta) \geq \sqrt{TR\varphi_\lambda(R)}$. Then we have

$$\begin{aligned}\int_{\Theta\in S^2} F(\Theta)dm_{S^2} &= \int_{\Theta\in U_{R,T}} F(\Theta)dm_{S^2} + \int_{\Theta\in U_{R,T}^c} F(\Theta)dm_{S^2} \\ &\geq m_{S^2}(U_{R,T})\sqrt{Q_-R\varphi_\lambda(R)} \\ &\quad +(4\pi - m_{S^2}(U_{R,T}))\sqrt{TR\varphi_\lambda(R)}. \end{aligned} \tag{6}$$

From inequalities (5) and (6), we have

$$\begin{aligned}4\pi\sqrt{2C_+R\varphi_\lambda(R)} &\geq m_{S^2}(U_{R,T})\sqrt{Q_-R\varphi_\lambda(R)} \\ &\quad +(4\pi - m_{S^2}(U_{R,T}))\sqrt{TR\varphi_\lambda(R)}.\end{aligned}$$

Thus we have

$$(\sqrt{T}-\sqrt{Q_-})m_{S^2}(U_{R,T})\sqrt{R\varphi_\lambda(R)} \geq 4\pi(\sqrt{T}-\sqrt{2C_+})\sqrt{R\varphi_\lambda(R)}.$$

Since $T > 2C_+ > Q_-$, we obtain

$$m_{S^2}(U_{R,T}) \geq 4\pi\frac{\sqrt{T}-\sqrt{2C_+}}{\sqrt{T}-\sqrt{Q_-}}.$$

□

Lemma 4.4. *For each $R \geq 0$ and $T > 0$, $\mu_\lambda^{-1}(B_{R,U_{R,T}})$ is a subset of $B_{g_\lambda}(p_0, \sqrt{\tau_{\lambda,T}(R)})$.*

Proof. First of all we take $p \in \mu_\lambda^{-1}(B_{R,U_{R,T}})$ such that $|\mu_\lambda(p)| > 1$. Since $\frac{\mu_\lambda(p)}{|\mu_\lambda(p)|}$ is an element of $U_{R,T}$, we have

$$L_\lambda + l_\lambda(\mu_\lambda(p)) \leq L_\lambda + l_\lambda\Big(R\frac{\mu_\lambda(p)}{|\mu_\lambda(p)|}\Big) \leq \sqrt{TR\varphi_\lambda(R)}.$$

from $R \geq |\mu_\lambda(p)| > 1$. Then we obtain $d_{g_\lambda}(p_0,p) \leq \sqrt{TR\varphi_\lambda(R)}$ from $d_{g_\lambda}(p_0,p) \leq L_\lambda + l_\lambda(\mu_\lambda(p))$.
If $p \in \mu_\lambda^{-1}(B_{R,U_{R,T}})$ is taken to be $|\mu_\lambda(p)| \leq 1$, then we have the same conclusion as above since $l_\lambda(\mu_\lambda(p)) = 0$ in this case. □

Now we fix a constant Q_+ to be $Q_+ > 2C_+$ and put $m_0 := 4\pi\frac{\sqrt{Q_+}-\sqrt{2C_+}}{\sqrt{Q_+}-\sqrt{Q_-}}$ and $P_- := m_0C_-$.

Proposition 4.5. *Let $P_-, Q_+ > 0$ be as above. Then we have*

$$\liminf_{r\to\infty} \frac{V_{g_\lambda}(p_0, r)}{\theta_{\lambda,P_-} \circ \tau^{-1}_{\lambda,Q_+}(r^2)} > 0.$$

Proof. Let $R \geq 0$. From Lemma 4.4, we have

$$V_{g_\lambda}(p_0, \sqrt{\tau_{\lambda,Q_+}(R)}) \geq \mathrm{vol}_{g_\lambda}(\mu_\lambda^{-1}(B_{R,U_{R,Q_+}})).$$

Then Proposition 4.1 gives

$$\begin{aligned} V_{g_\lambda}(p_0, \sqrt{\tau_{\lambda,Q_+}(R)}) &\geq m_{S^2}(U_{R,Q_+})C_- R^2 \varphi_\lambda(R) \\ &\geq m_0 C_- R^2 \varphi_\lambda(R) = \theta_{\lambda,P_-}(R) \end{aligned}$$

for $R \geq R_0$. Thus we have the assertion by putting $R = \tau^{-1}_{\lambda,Q_+}(r^2)$. □

5 The volume growth

In Sections 3 and 4, we estimate $V_{g_\lambda}(p_0, r)$ from the above by $\theta_{\lambda,P_+} \circ \tau^{-1}_{\lambda,Q_-}(r^2)$ and from the bottom by $\theta_{\lambda,P_-} \circ \tau^{-1}_{\lambda,Q_+}(r^2)$. In this section we show that the asymptotic behavior of the functions $\theta_{\lambda,P_+} \circ \tau^{-1}_{\lambda,Q_-}(r^2)$ and $\theta_{\lambda,P_-} \circ \tau^{-1}_{\lambda,Q_+}(r^2)$ are equal up to constant, and prove the main results.
The asymptotic behavior of $V_{g_\lambda}(p, r)$ is independent of the choice of $p \in X_\lambda$ from the next well-known fact.

Proposition 5.1. *Let (X, g) be a connected Riemannian manifold of dimension n, whose Ricci curvature is nonnegative. Then we have*

$$\lim_{r\to+\infty} \frac{V_g(p_1, r)}{V_g(p_0, r)} = 1$$

for any $p_0, p_1 \in X$.

Proof. From the Bishop-Gromov comparison inequality $\frac{V_g(p_1,r)}{r^n}$ is nonincreasing with respect to r. If we put $r_0 := d_g(p_0, p_1)$, then we have

$$\begin{aligned} \frac{V_g(p_1, r)}{V_g(p_0, r)} &\leq \frac{V_g(p_0, r + r_0)}{V_g(p_0, r)} \\ &= \frac{r^n}{V_g(p_0, r)} \frac{V_g(p_0, r + r_0)}{(r + r_0)^n} \frac{(r + r_0)^n}{r^n} \\ &\leq \frac{r^n}{V_g(p_0, r)} \frac{V_g(p_0, r)}{r^n} \frac{(r + r_0)^n}{r^n} \\ &= \frac{(r + r_0)^n}{r^n} \to 1 \quad (r \to +\infty). \end{aligned}$$

By considering the same argument after exchanging p_0 and p_1, we have the assertion. $\square$

We denote by C^0_+ the set of the nondecreasing continuous functions from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$.

Definition 5.2. For $f_0, f_1 \in C^0_+$, we define $f_0(r) \preceq_r f_1(r)$ if

$$\limsup_{r\to+\infty} \frac{f_0(r)}{f_1(r)} < +\infty.$$

Definition 5.3. For $f_0, f_1 \in C^0_+$, we define $f_0(r) \simeq_r f_1(r)$ if $f_0(r) \preceq_r f_1(r)$ and $f_1(r) \preceq_r f_0(r)$.

For a Riemannian manifold (X, g) and a point $p_0 \in X$, the function $V_g(p_0, r)$ is an element of C^0_+. If (X, g) satisfies the assumption of Proposition 5.1, then the equivalence class of $V_g(p_0, r)$ with respect to $\simeq_r$ is independent of the choice of $p_0 \in X$. Therefore we denoted by $V_g(r)$ the equivalence class of $V_g(p_0, r)$. For example, if we write $V_g(r) \preceq_r r^n$, then it implies that

$$\limsup_{r\to+\infty} \frac{V_g(p_0, r)}{r^n} < +\infty$$

for some (hence all) $p_0 \in X$.
The main purpose of this section is to look for the function in C^0_+ which is equivalent to $V_{g_\lambda}(r)$.

Lemma 5.4. *Let $S_+, S_-, T_+, T_- > 0$. Then $\theta_{\lambda,S_+} \circ \tau^{-1}_{\lambda,T_-}$ and $\theta_{\lambda,S_-} \circ \tau^{-1}_{\lambda,T_+}$ are elements of C^0_+ and we have $\theta_{\lambda,S_+} \circ \tau^{-1}_{\lambda,T_-}(r^2) \simeq_r \theta_{\lambda,S_-} \circ \tau^{-1}_{\lambda,T_+}(r^2)$.*

Proof. We may suppose $T_- \leq T_+$ without loss of generality. Since τ_{λ,T_+} and τ_{λ,T_-} are continuous, strictly increasing and satisfy $\tau_{\lambda,T_+}(0) = \tau_{\lambda,T_-}(0) = 0$, then τ^{-1}_{λ,T_+} and τ^{-1}_{λ,T_-} are also the elements of C^0_+. Hence the composite functions $\theta_{\lambda,S_+} \circ \tau^{-1}_{\lambda,T_-}$ and $\theta_{\lambda,S_-} \circ \tau^{-1}_{\lambda,T_+}$ are also the elements of C^0_+.
Next we show (i) $\theta_{\lambda,S_-} \circ \tau^{-1}_{\lambda,T_+}(r^2) \preceq_r \theta_{\lambda,S_+} \circ \tau^{-1}_{\lambda,T_-}(r^2)$ and (ii) $\theta_{\lambda,S_+} \circ \tau^{-1}_{\lambda,T_-}(r^2) \preceq_r \theta_{\lambda,S_-} \circ \tau^{-1}_{\lambda,T_+}(r^2)$.
(i) From $\tau_{\lambda,T_+}(R) \geq \tau_{\lambda,T_-}(R)$ and strictly increasingness of $\tau_{\lambda,T_\pm}$, then we have $\tau^{-1}_{\lambda,T_+}(r^2) \leq \tau^{-1}_{\lambda,T_-}(r^2)$. Hence we obtain

$$\frac{\theta_{\lambda,S_-} \circ \tau^{-1}_{\lambda,T_+}(r^2)}{\theta_{\lambda,S_+} \circ \tau^{-1}_{\lambda,T_-}(r^2)} \leq \frac{\theta_{\lambda,S_-} \circ \tau^{-1}_{\lambda,T_-}(r^2)}{\theta_{\lambda,S_+} \circ \tau^{-1}_{\lambda,T_-}(r^2)} \leq \frac{S_-}{S_+}.$$

(ii) Put $R_\pm := \tau^{-1}_{\lambda,T_\pm}(r^2)$. Then we have

$$r^2 = T_+ R_+ \varphi_\lambda(R_+) = T_- R_- \varphi_\lambda(R_-).$$

Since φ_λ is nondecreasing, it holds

$$R_-\varphi_\lambda(R_-) = \frac{T_+}{T_-}R_+\varphi_\lambda(R_+) \leq \frac{T_+}{T_-}R_+\varphi_\lambda\Big(\frac{T_+}{T_-}R_+\Big) \tag{7}$$

Since the function $R\varphi_\lambda(R)$ is strictly increasing with respect to R, the expression (7) gives $R_- \leq \frac{T_+}{T_-}R_+$. Recall that φ_λ satisfies $\varphi_\lambda(\alpha R) \leq \alpha\varphi_\lambda(R)$ for $\alpha \geq 1$, which implies $\theta_{\lambda,S_+}(\alpha R) \leq \alpha^3\theta_{\lambda,S_+}(R)$. Thus we have

$$\begin{aligned}\frac{\theta_{\lambda,S_+}\circ\tau^{-1}_{\lambda,T_-}(r^2)}{\theta_{\lambda,S_-}\circ\tau^{-1}_{\lambda,T_+}(r^2)} = \frac{\theta_{\lambda,S_+}(R_-)}{\theta_{\lambda,S_-}(R_+)} \leq \frac{\theta_{\lambda,S_+}\left(\frac{T_+}{T_-}R_+\right)}{\theta_{\lambda,S_-}(R_+)} &\leq \Big(\frac{T_+}{T_-}\Big)^3\frac{\theta_{\lambda,S_+}(R_+)}{\theta_{\lambda,S_-}(R_+)} \\ &= \Big(\frac{T_+}{T_-}\Big)^3\frac{S_+}{S_-}.\end{aligned}$$

□

Put $\theta_\lambda := \theta_{\lambda,1}$, $\tau_\lambda := \tau_{\lambda,1}$. Then the main result in this paper is described as follows.

Theorem 5.5. *For each $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$ and $p \in X_\lambda$, the function $V_{g_\lambda}(p,r)$ satisfies*

$$0 < \liminf_{r\to+\infty}\frac{V_{g_\lambda}(p,r)}{r^2\tau_\lambda^{-1}(r^2)} \leq \limsup_{r\to+\infty}\frac{V_{g_\lambda}(p,r)}{r^2\tau_\lambda^{-1}(r^2)} < +\infty.$$

Proof. We have shown that

$$\theta_{\lambda,P_-}\circ\tau^{-1}_{\lambda,Q_+}(r^2) \preceq_r V_{g_\lambda}(r) \preceq_r \theta_{\lambda,P_+}\circ\tau^{-1}_{\lambda,Q_-}(r^2)$$

in Sections 3 and 4, and

$$\theta_{\lambda,P_-}\circ\tau^{-1}_{\lambda,Q_+}(r^2) \simeq_r \theta_{\lambda,P_+}\circ\tau^{-1}_{\lambda,Q_-}(r^2) \simeq_r \theta_\lambda\circ\tau_\lambda^{-1}(r^2)$$

in Lemma 5.4. Thus we have the assertion from

$$\theta_\lambda\circ\tau_\lambda^{-1}(r^2) = \tau_\lambda^{-1}(r^2)\cdot\tau_\lambda(\tau_\lambda^{-1}(r^2)) = r^2\tau_\lambda^{-1}(r^2).$$

□

Corollary 5.6. *For $\lambda, \lambda' \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$, the condition $V_{g_\lambda}(r) \preceq_r V_{g_{\lambda'}}(r)$ is equivalent to $\varphi_\lambda(R) \preceq_R \varphi_{\lambda'}(R)$.*

Proof. The condition $V_{g_\lambda}(r) \preceq_r V_{g_{\lambda'}}(r)$ is equivalent to $\tau_\lambda^{-1}(r^2) \preceq_r \tau_{\lambda'}^{-1}(r^2)$ from Theorem 5.5.
If we assume $\tau_\lambda^{-1}(r^2) \preceq_r \tau_{\lambda'}^{-1}(r^2)$ then there are constants $r_0 \geq 0$ and $C \geq 1$ which satisfy $\frac{\tau_\lambda^{-1}(r^2)}{\tau_{\lambda'}^{-1}(r^2)} \leq C$ for all $r \geq r_0$. Now we put $r^2 := \tau_\lambda(R)$ and $(r')^2 := \tau_{\lambda'}(R)$ for $R \geq \tau_\lambda^{-1}(r_0^2)$. Since we have

$$R = \tau_{\lambda'}^{-1}((r')^2) = \tau_\lambda^{-1}(r^2) \leq C\tau_{\lambda'}^{-1}(r^2),$$

then

$$(r')^2 \leq \tau_{\lambda'}(C\tau_{\lambda'}^{-1}(r^2)) \leq C^2\tau_{\lambda'}(\tau_{\lambda'}^{-1}(r^2)) = C^2r^2$$

is given by the monotonicity of $\tau_{\lambda'}$. Thus we obtain $\frac{\tau_{\lambda'}(R)}{\tau_\lambda(R)} \leq C^2$ for all $R \geq \tau_{\lambda'}^{-1}(r_0^2)$, which implies $\tau_{\lambda'}(R) \preceq_R \tau_\lambda(R)$.
On the other hand, if we assume $\tau_{\lambda'}(R) \preceq_R \tau_\lambda(R)$ then $\tau_\lambda^{-1}(r^2) \preceq_r \tau_{\lambda'}^{-1}(r^2)$ is obtained in the same way. Thus we have the assertion since the condition $\tau_{\lambda'}(R) \preceq_R \tau_\lambda(R)$ is equivalent to $\varphi_{\lambda'}(R) \preceq_R \varphi_\lambda(R)$. ☐

Lemma 5.7. *For all* $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$, *we have*

$$\lim_{R\to+\infty} \varphi_\lambda(R) = +\infty.$$

Proof. From Lemma 3.2, it is enough to show $\lim_{R\to+\infty} \psi_\lambda(R) = +\infty$, which follows directly from $\lim_{R\to+\infty} \sharp N_\lambda(R) = +\infty$. ☐

Lemma 5.8. *For all* $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$, *we have*

$$\lim_{R\to+\infty} \frac{\varphi_\lambda(R)}{R} = 0.$$

Proof. For each $\varepsilon > 0$ there exists a sufficiently large $n_\varepsilon \in \mathbb{Z}_{>0}$ such that $\sum_{|n|>n_\varepsilon} \frac{1}{|\lambda_n|} < \frac{\varepsilon}{2}$. Hence we have

$$\begin{aligned}
\frac{\varphi_\lambda(R)}{R} &= \sum_{|n|\leq n_\varepsilon} \frac{1}{R+|\lambda_n|} + \sum_{|n|>n_\varepsilon} \frac{1}{R+|\lambda_n|} \\
&\leq \sum_{|n|\leq n_\varepsilon} \frac{1}{R} + \sum_{|n|>n_\varepsilon} \frac{1}{|\lambda_n|} \\
&\leq \frac{2n_\varepsilon+1}{R} + \frac{\varepsilon}{2}.
\end{aligned}$$

Then the inequality $\frac{\varphi_\lambda(R)}{R} \leq \varepsilon$ holds for any $R \geq \frac{2(2n_\varepsilon+1)}{\varepsilon}$, which implies $\lim_{R\to+\infty} \frac{\varphi_\lambda(R)}{R} \leq \varepsilon$. The assertion follows by taking $\varepsilon \to 0$. ☐

Corollary 5.9. *For all $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$, we have*

$$\lim_{r\to+\infty} \frac{V_{g_\lambda}(r)}{r^4} = 0, \quad \lim_{r\to+\infty} \frac{V_{g_\lambda}(r)}{r^3} = +\infty.$$

Proof. It suffices to show that

$$\lim_{r\to+\infty} \frac{\theta_\lambda \circ \tau_\lambda^{-1}(r^2)}{r^4} = 0, \quad \lim_{r\to+\infty} \frac{\theta_\lambda \circ \tau_\lambda^{-1}(r^2)}{r^3} = +\infty.$$

We put $R = \tau_\lambda^{-1}(r^2)$ and consider the limit of $R \to +\infty$. Then we have

$$\frac{\theta_\lambda \circ \tau_\lambda^{-1}(r^2)}{r^4} = \frac{\theta_\lambda(R)}{\tau_\lambda(R)^2} = \frac{1}{\varphi_\lambda(R)},$$

and

$$\frac{\theta_\lambda \circ \tau_\lambda^{-1}(r^2)}{r^3} = \frac{\theta_\lambda(R)}{\tau_\lambda(R)^{\frac{3}{2}}} = \sqrt{\frac{R}{\varphi_\lambda(R)}}.$$

Hence we obtain

$$\begin{aligned}
\lim_{r\to+\infty} \frac{\theta_\lambda \circ \tau_\lambda^{-1}(r^2)}{r^4} &= \lim_{R\to+\infty} \frac{1}{\varphi_\lambda(R)} = 0,\\
\lim_{r\to+\infty} \frac{\theta_\lambda \circ \tau_\lambda^{-1}(r^2)}{r^3} &= \lim_{R\to+\infty} \sqrt{\frac{R}{\varphi_\lambda(R)}} = +\infty.
\end{aligned}$$

from Lemmas 5.7 and 5.8. □

The condition $\varphi_\lambda(R) \preceq_R \varphi_{\lambda'}(R)$ is rather difficult to check. But we can describe the sufficient condition for $\varphi_\lambda(R) \preceq_R \varphi_{\lambda'}(R)$ easier as follows.

Proposition 5.10. *Let $\lambda, \lambda' \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$. Suppose that there are some $\alpha > 0$ and $R_0 > 0$ such that $\sharp N_\lambda(R) \le \sharp N_{\alpha\lambda'}(R)$ for $R \ge R_0$, where $\alpha\lambda' := (\alpha\lambda'_n)_{n\in\mathbb{Z}}$. Then we have $\varphi_\lambda(R) \preceq_R \varphi_{\lambda'}(R)$.*

Proof. Take bijections $a, b : \mathbb{Z}_{>0} \to \mathbb{Z}$ to be $|\lambda_{a(n)}| \le |\lambda_{a(n+1)}|$, $|\lambda'_{b(n)}| \le |\lambda'_{b(n+1)}|$ for each $n \in \mathbb{Z}_{>0}$. Then the condition $\sharp N_\lambda(R) \le \sharp N_{\alpha\lambda'}(R)$ is equivalent to $a^{-1}(N_\lambda(R)) \subset b^{-1}(N_{\alpha\lambda'}(R))$.
Take $n_0 \in \mathbb{Z}_{>0}$ sufficiently large such that $|\lambda_{a(n_0)}| \ge R_0$. Then we have $|\alpha\lambda'_{b(n)}| \le |\lambda_{a(n)}|$ for each $n \ge n_0$ from

$$a^{-1}(N_\lambda(|\lambda_{a(n)}|)) \subset b^{-1}(N_{\alpha\lambda'}(|\lambda_{a(n)}|)).$$

If n is an element of $b^{-1}(N_{\alpha\lambda'}(R))\backslash a^{-1}(N_\lambda(R))$ for $R \geq R_0$, then we have

$$\frac{1}{|\lambda_{a(n)}|} \leq \frac{1}{R}, \quad \frac{1}{|\lambda_{a(n)}|} \leq \frac{1}{\alpha|\lambda'_{b(n)}|}.$$

Thus we have

$$\begin{aligned}
\psi_\lambda(R) &= \sharp N_\lambda(R) + \sum_{n \notin a^{-1}(N_\lambda(R))} \frac{R}{|\lambda_{a(n)}|} \\
&= \sharp N_\lambda(R) + \sum_{n \in b^{-1}(N_{\alpha\lambda'}(R))\backslash a^{-1}(N_\lambda(R)^c)} \frac{R}{|\lambda_{a(n)}|} + \sum_{n \notin b^{-1}(N_{\alpha\lambda'}(R))} \frac{R}{|\lambda_{a(n)}|} \\
&\leq \sharp N_\lambda(R) + (\sharp N_{\alpha\lambda'}(R) - \sharp N_\lambda(R)) + \sum_{n \notin b^{-1}(N_{\alpha\lambda'}(R))} \frac{R}{\alpha|\lambda'_{b(n)}|} \\
&= \psi_{\alpha\lambda'}(R)
\end{aligned}$$

for $R \geq R_0$. From $\psi_{\lambda'}(R) \simeq_R \psi_{\alpha\lambda'}(R)$, we have $\psi_\lambda(R) \preceq_R \psi_{\lambda'}(R)$ which is equivalent to $\varphi_\lambda(R) \preceq_R \varphi_{\lambda'}(R)$. □

6 Examples

In this section we evaluate $V_{g_\lambda}(r)$ concretely for some $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$. We fix a bijection $a : \mathbb{Z}_{>0} \to \mathbb{Z}$ and identify $\mathbb{Z}$ with $\mathbb{Z}_{>0}$ throughout this section. Now we fix $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$. Assume that $|\lambda_{a(n)}|$ is nondecreasing with respect to $n \in \mathbb{Z}_{>0}$ and there exists a continuous nondecreasing function $\lambda_{\mathbb{R}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which satisfies $|\lambda_{a(n)}| = \lambda_{\mathbb{R}}(n)$. Then a function $\hat{\varphi}_{\lambda_{\mathbb{R}}}(R) := \int_0^\infty \frac{Rdx}{R+\lambda_{\mathbb{R}}(x)}$ is strictly increasing and satisfies $\hat{\varphi}_{\lambda_{\mathbb{R}}}(R) \simeq_R \varphi_\lambda(R)$. In this case we have $V_{g_\lambda}(r) \simeq_r r^2 \hat{\tau}_{\lambda_{\mathbb{R}}}^{-1}(r^2)$ where $\hat{\tau}_{\lambda_{\mathbb{R}}}$ is defined by $\hat{\tau}_{\lambda_{\mathbb{R}}}(R) := R\hat{\varphi}_{\lambda_{\mathbb{R}}}(R)$. Now we compute the volume growth of g_λ in the following two cases.

1. Fix $\alpha > 1$ and put $\lambda_{\mathbb{R}}(x) = x^\alpha$, $\lambda_{a(n)} = \lambda_{\mathbb{R}}(n)i$. Then $\hat{\varphi}_{\lambda_{\mathbb{R}}}$ is given by

$$\hat{\varphi}_{\lambda_{\mathbb{R}}}(R) = \int_0^\infty \frac{Rdx}{R + x^\alpha} = R^{\frac{1}{\alpha}} \int_0^\infty \frac{dy}{1 + y^\alpha},$$

where we put $y = \frac{x}{R^{\frac{1}{\alpha}}}$. Since $\int_0^\infty \frac{dy}{1+y^\alpha}$ is a constant, we have $\hat{\varphi}_{\lambda_{\mathbb{R}}}(R) \simeq_R R^{\frac{1}{\alpha}}$, which gives $\hat{\tau}_{\lambda_{\mathbb{R}}}(R) \simeq_R R^{1+\frac{1}{\alpha}}$ and $\hat{\tau}_{\lambda_{\mathbb{R}}}^{-1}(r^2) \simeq_r r^{\frac{2\alpha}{\alpha+1}}$. Hence the volume growth is given by

$$V_{g_\lambda}(r) \simeq_r r^{4-\frac{2}{\alpha+1}}.$$

Thus we obtain the following result.

Theorem 6.1. *There exists a complete 4-dimensional hyperkähler manifold (X_λ, g_λ) for each $3 < \alpha < 4$ whose volume growth is given by*

$$V_{g_\lambda}(r) \simeq_r r^\alpha.$$

2. Fix $\alpha > 0$ and put $\lambda_{\mathbb{R}}(x) = e^{\alpha x}$, $\lambda_{a(n)} = \lambda_{\mathbb{R}}(n)i$. Then $\hat{\varphi}_{\lambda_{\mathbb{R}}}$ is given by

$$\hat{\varphi}_{\lambda_{\mathbb{R}}}(R) = \int_0^\infty \frac{Rdx}{R + e^{\alpha x}}.$$

By putting $y = e^{\alpha x}$, we have

$$\begin{aligned}\hat{\varphi}_{\lambda_{\mathbb{R}}}(R) &= \int_1^\infty \frac{Rdy}{\alpha y(y+R)} \\ &= \frac{1}{\alpha}\log(R+1).\end{aligned}$$

Hence we have $\hat{\varphi}_{\lambda_{\mathbb{R}}}(R) = \frac{1}{\alpha}\log(R+1)$ and $\hat{\tau}_{\lambda_{\mathbb{R}}}(R) = \frac{1}{\alpha}R\log(R+1)$.

Proposition 6.2. *Let $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$ be as above. Then the volume growth of g_λ satisfies $V_{g_\lambda}(r) \simeq_r \frac{r^4}{\log r}$.*

Proof. It is enough to see the behavior of $\frac{r^2\hat{\tau}_{\lambda_{\mathbb{R}}}^{-1}(r^2)\log r}{r^4}$ at $r \to +\infty$. Put $R = \hat{\tau}_{\lambda_{\mathbb{R}}}^{-1}(r^2)$. Then we have $r^2 = \frac{1}{\alpha}R\log(R+1)$ and $\log r = \frac{1}{2}(\log R + \log\log(R+1) - \log\alpha)$. Thus we have

$$\begin{aligned}\lim_{r\to+\infty}\frac{r^2\hat{\tau}_{\lambda_{\mathbb{R}}}^{-1}(r^2)\log r}{r^4} &= \lim_{R\to+\infty}\frac{\alpha}{2}\frac{R(\log R + \log\log(R+1) - \log\alpha)}{R\log(R+1)} \\ &= \frac{\alpha}{2}.\end{aligned}$$

□

Thus we have the following theorem.

Theorem 6.3. *There exists a complete 4-dimensional hyperkähler manifold (X_λ, g_λ) whose volume growth satisfies*

$$\lim_{r\to+\infty}\frac{V_{g_\lambda}(r)}{r^4} = 0, \quad \lim_{r\to+\infty}\frac{V_{g_\lambda}(r)}{r^\alpha} = +\infty$$

for any $\alpha < 4$.

7 Taub-NUT deformations

We consider the volume growth of the Taub-NUT deformations of (X_λ, g_λ) in this section.
First of all, we define the Taub-NUT deformations of hyperkähler manifolds with tri-Hamiltonian S^1-actions. Let (X, ω) be a hyperkähler manifold of dimension 4 with tri-Hamiltonian S^1-action, and $\mu : X \to \mathrm{Im}\mathbb{H}$ be the hyperkähler moment map. An action of $\mathbb{R}$ on $\mathbb{H}$ is defined by $x \mapsto x + \sqrt{s}^{-1}t$ for $x \in \mathbb{H}$ and $t \in \mathbb{R}$ by fixing a constant $s > 0$. This action preserves the standard hyperkähler structure on $\mathbb{H}$ and the hyperkähler moment map is given by $\sqrt{s}^{-1} \cdot Im : \mathbb{H} \to \mathrm{Im}\mathbb{H}$. Then we obtain the hyperkähler quotient with respect to the action of $\mathbb{R}$ on $X \times \mathbb{H}$, that is, the quotient $(\mu^{(s)})^{-1}(\zeta)/\mathbb{R}$ where ζ is an element of $\mathrm{Im}\mathbb{H}$ and $\mu^{(s)} : X \times \mathbb{H} \to \mathrm{Im}\mathbb{H}$ is defined by $\mu^{(s)}(x, y) := \mu(x) + 2\sqrt{s}^{-1} Im(y)$. The hyperkähler structure on $(\mu^{(s)})^{-1}(\zeta)/\mathbb{R}$ is independent of ζ.
For each $\zeta \in \mathrm{Im}\mathbb{H}$ we have an imbedding $\tilde{\iota}_{s,\zeta} : X \to (\mu^{(s)})^{-1}(\zeta)$ defined by $\tilde{\iota}_{s,\zeta}(x) := (x, \frac{\sqrt{s}}{2}(-\mu(x) + \zeta))$ which induces a diffeomorphism $\iota_{s,\zeta} : X \to (\mu^{(s)})^{-1}(\zeta)/\mathbb{R}$. Then we have another hyperkähler structure on X independent of ζ by the pull-back, which is called Taub-NUT deformation of ω denoted by $\omega^{(s)}$. If we denote by g the hyperkähler metric of (X, ω), then we denote by $g^{(s)}$ the hyperkähler metric of $(X, \omega^{(s)})$.
There is the pair of a harmonic function and an S^1-connection (Φ, A) corresponding to the hyperkähler structure ω by Theorem 2.9. Then the corresponding pair to $\omega^{(s)}$ is given by $(\Phi + \frac{s}{4}, A)$.
The S^1-action on X also preserves $\omega^{(s)}$ and $\mu : X \to \mathrm{Im}\mathbb{H}$ is also the hyperkähler moment map with respect to $\omega^{(s)}$.

Lemma 7.1. *Let (X, g) be as above and take $p_0, p \in X$. We suppose that p_0 is a fixed point by the S^1-action. Then we have the inequality*

$$d_{g^{(s)}}(p_0, p)^2 \geq d_g(p_0, p)^2 + \frac{s}{4}|\mu(p) - \mu(p_0)|^2.$$

Proof. We apply the same argument as Proposition 3.1. Then the assertion follows from

$$\begin{aligned} d_{g^{(s)}}(p_0, p)^2 &\geq \inf_{t\in\mathbb{R}} d_{g\times g_{\mathbb{H}}}(\tilde{\iota}_{s,\zeta}(p_0)\cdot t, \tilde{\iota}_{s,\zeta}(p))^2 \\ &= \inf_{t\in\mathbb{R}} (d_g(p_0 e^{it}, p)^2 + |\frac{\sqrt{s}}{2}(\mu(p_0) - \mu(p)) + \frac{1}{\sqrt{s}}t|^2) \\ &= \inf_{t\in\mathbb{R}} \left(d_g(p_0, p)^2 + \frac{s}{4}|\mu(p_0) - \mu(p)|^2 + \frac{t^2}{s} \right) \\ &= d_g(p_0, p)^2 + \frac{s}{4}|\mu(p_0) - \mu(p)|^2, \end{aligned}$$

where $g_{\mathbb{H}}$ is the Euclidean metric on $\mathbb{H}$ and $g \times g_{\mathbb{H}}$ is the direct product metric. $\square$

Lemma 7.2. *Let (X, g) be as above and $B \subset \mathrm{Im}\mathbb{H}$ be a measurable set. Then we have*

$$\mathrm{vol}_{g^{(s)}}(\mu^{-1}(B)) = \mathrm{vol}_g(\mu^{-1}(B)) + \frac{\pi s}{4} m_{\mathrm{Im}\mathbb{H}}(B),$$

where $m_{\mathrm{Im}\mathbb{H}}$ is the Lebesgue measure of $\mathrm{Im}\mathbb{H}$.

Proof. It follows directly from Lemma 2.10 and that $\omega^{(s)}$ corresponds to $(\Phi + \frac{s}{4}, A)$. $\square$

For $s, C > 0$ and $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$, put

$$\theta_{\lambda,C}^{(s)}(R) := CR^2\varphi_\lambda(R) + \frac{\pi^2 s}{3}R^3, \quad \tau_{\lambda,C}^{(s)}(R) := CR\varphi_\lambda(R) + \frac{s}{4}R^2.$$

Proposition 7.3. *For $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$ and $s > 0$, we have*

$$\limsup_{r \to +\infty} \frac{V_{g_\lambda^{(s)}}(p_0, r)}{r^3} \le \frac{8\pi^2}{3\sqrt{s}}.$$

Proof. From Lemma 7.1 and 7.2, we have

$$V_{g_\lambda^{(s)}}(p_0, r) \le \theta_{\lambda,P_+}^{(s)} \circ (\tau_{\lambda,Q_-}^{(s)})^{-1}(r^2)$$

for $r > 0$. Then it suffices to show

$$\limsup_{r \to +\infty} \frac{\theta_{\lambda,P_+}^{(s)} \circ (\tau_{\lambda,Q_-}^{(s)})^{-1}(r^2)}{r^3} \le \frac{8\pi^2}{3\sqrt{s}}.$$

Put $R = (\tau_{\lambda,Q_-}^{(s)})^{-1}(r^2)$. Then we have

$$r^2 = Q_- R\varphi_\lambda(R) + \frac{s}{4}R^2 \ge \frac{s}{4}R^2.$$

Therefore Lemma 5.8 gives

$$\limsup_{r \to +\infty} \frac{\theta_{\lambda,P_+}^{(s)} \circ (\tau_{\lambda,Q_-}^{(s)})^{-1}(r^2)}{r^3} \le \limsup_{R \to +\infty} \frac{8(P_+ R^2 \varphi_\lambda(R) + \frac{\pi^2 s}{3}R^3)}{\sqrt{s}^3 R^3} = \frac{8\pi^2}{3\sqrt{s}}.$$

$\square$

Next we consider the lower bound for $V_{g_\lambda^{(s)}}(r)$. We apply the same way as Section 4. Put $l_\lambda^{(s)} : \mathrm{Im}\mathbb{H} \to \mathbb{R}$ as

$$l_\lambda^{(s)}(\zeta) := \int_1^{|\zeta|} \sqrt{\Phi_\lambda\Big(t\frac{\zeta}{|\zeta|}\Big) + \frac{s}{4}}\, dt$$

on $|\zeta| > 1$, and $l_\lambda^{(s)}(\zeta) := 0$ on $|\zeta| \le 1$. Then the inequality $d_{g_\lambda^{(s)}}(p_0, p) \le L_\lambda^{(s)} + l_\lambda^{(s)}(\mu_\lambda(p))$ holds where $p_0 \in X$ is taken as in Sections 3 and 4, and we put $L_\lambda^{(s)} := \sup_{p\in\mu_\lambda^{-1}(\bar{B}_1)} d_{g_\lambda^{(s)}}(p_0, p)$.

Lemma 7.4. *Let $C_+ > 0$ be as in Proposotion 4.2. Then we have*

$$\int_{\Theta\in S^2} l_\lambda^{(s)}(R\Theta)dm_{S^2} \le 4\pi(\sqrt{\tau_{\lambda,C_+}(R)} + \sqrt{\frac{sR^2}{4}}).$$

Proof. The assertion follows from

$$\begin{aligned}\int_{\Theta\in S^2}\Big(\int_1^R \sqrt{\Phi_\lambda(t\Theta) + \frac{s}{4}}\, dt\Big)dm_{S^2} \;&\le\; \int_{\Theta\in S^2}\Big(\int_1^R \sqrt{\frac{s}{4}}dt\Big)dm_{S^2} \\ &\quad + \int_{\Theta\in S^2} l_\lambda(R\Theta)dm_{S^2} \\ &\le\; 4\pi\sqrt{\frac{sR^2}{4}} + 4\pi\sqrt{\tau_{\lambda,C_+}(R)}.\end{aligned}$$

□

Put $U_{R,T}^{(s)} := \{\Theta \in S^2; L_\lambda^{(s)} + l_\lambda^{(s)}(R\Theta) \le \sqrt{\tau_{\lambda,T}(R)} + \sqrt{\frac{sR^2}{4}}\}$.

Lemma 7.5. *There is a constant $R_0 > 0$ such that*

$$m_{S^2}(U_{R,T}^{(s)}) \ge \frac{4\pi(\sqrt{T} - \sqrt{2C_+})}{\sqrt{T}}$$

for any $R \ge R_0$ and $T > 2C_+$.

Proof. We consider the same argument as in Lemma 4.3. First of all we remark that there exists sufficiently large $R_0 > 0$ such that

$$\int_{\Theta\in S^2}(L_\lambda^{(s)} + l_\lambda^{(s)}(R\Theta))dm_{S^2} \le 4\pi(\sqrt{\tau_{\lambda,2C_+}(R)} + \sqrt{\frac{sR^2}{4}})$$

for any $R \geq R_0$. Then we have

$$\begin{aligned}
\int_{\Theta \in S^2} (L_\lambda^{(s)} + l_\lambda^{(s)}(R\Theta)) dm_{S^2} &= \int_{\Theta \in U_{R,T}^{(s)}} (L_\lambda^{(s)} + l_\lambda^{(s)}(R\Theta)) dm_{S^2} \\
&\quad + \int_{\Theta \in S^2 \setminus U_{R,T}^{(s)}} (L_\lambda^{(s)} + l_\lambda^{(s)}(R\Theta)) dm_{S^2} \\
&\geq m_{S^2}(U_{R,T}^{(s)}) \sqrt{\tau_{\lambda,Q_-}(R) + \frac{sR^2}{4}} \\
&\quad + (4\pi - m_{S^2}(U_{R,T}^{(s)}))(\sqrt{\tau_{\lambda,T}(R)} + \sqrt{\frac{sR^2}{4}}),
\end{aligned}$$

where the constant $Q_- > 0$ is as in Proposition 3.1. Then an inequality $\sqrt{\tau_{\lambda,Q_-}(R) + \frac{sR^2}{4}} \geq \sqrt{\frac{sR^2}{4}}$ gives

$$\begin{aligned}
\int_{\Theta \in S^2} (L_\lambda^{(s)} + l_\lambda^{(s)}(R\Theta)) dm_{S^2} &\geq 4\pi \sqrt{\frac{sR^2}{4}} \\
&\quad + (4\pi - m_{S^2}(U_{R,T}^{(s)})) \sqrt{\tau_{\lambda,T}(R)}.
\end{aligned}$$

Thus we have the conclusion. $\square$

Lemma 7.6. *For each $R \geq 0$ and $T > 0$, $\mu_\lambda^{-1}(B_{R,U_{R,T}^{(s)}})$ is a subset of $B_{g_\lambda^{(s)}}\left(p_0, \sqrt{\tau_{\lambda,T}(R)} + \sqrt{\frac{sR^2}{4}}\right)$.*

Proof. It is shown by the same argument as Lemma 4.4. $\square$

Proposition 7.7. *For each $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$ and $s > 0$ we have*

$$\liminf_{r \to +\infty} \frac{V_{g_\lambda^{(s)}}(p_0, r)}{r^3} \geq \frac{8\pi^2}{3\sqrt{s}}.$$

Proof. For each sufficiently small $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that

$$m_{S^2}(U_{R,T_\varepsilon}^{(s)}) > 4\pi(1 - \varepsilon)$$

from Lemma 7.5. Since $\frac{\varphi_\lambda(R)}{R}$ converges to 0, there exists $R_\varepsilon > 0$ such that $T_\varepsilon \varphi_\lambda(R) < \varepsilon^2 R$ for any $R > R_\varepsilon$. Then it holds

$$\begin{aligned}
\mathrm{vol}_{g_\lambda^{(s)}}\left(\mu_\lambda^{-1}\left(B_{R,U_{R,T_\varepsilon}^{(s)}}\right)\right) &\leq V_{g_\lambda^{(s)}}\left(p_0, \sqrt{\tau_{\lambda,T_\varepsilon}(R)} + \sqrt{\frac{sR^2}{4}}\right) \\
&\leq V_{g_\lambda^{(s)}}\left(p_0, (\varepsilon + \sqrt{\frac{s}{4}})R\right) \qquad (8)
\end{aligned}$$

for $R > R_\varepsilon$. On the other hand, we have the following inequality

$$\mathrm{vol}_{g_\lambda^{(s)}}(\mu_\lambda^{-1}(B_{R,U})) \geq C_- m_{S^2}(U) R^2 \cdot \varphi_\lambda(R) + \frac{\pi m_{S^2}(U)}{12} s R^3 \tag{9}$$

for $U \subset \mathrm{Im}\mathbb{H}$ from Proposition 4.1. Thus we obtain

$$\begin{aligned} V_{g_\lambda^{(s)}}\Big(p_0, \Big(\varepsilon + \sqrt{\frac{s}{4}}\Big) R\Big) &\geq C_- m_{S^2}(U_{R,T_\varepsilon}^{(s)}) R^2 \cdot \varphi_\lambda(R) + \frac{\pi m_{S^2}(U_{R,T_\varepsilon}^{(s)})}{12} s R^3 \\ &\geq \frac{\pi^2}{3}(1-\varepsilon) s R^3 \end{aligned}$$

from (8) and (9). Hence substituting $r = (\varepsilon + \sqrt{\frac{s}{4}})R$ gives

$$\liminf_{r\to+\infty} \frac{V_{g_\lambda^{(s)}}(p_0, r)}{r^3} = \liminf_{R\to+\infty} \frac{V_{g_\lambda^{(s)}}(p_0, (\varepsilon + \sqrt{\frac{s}{4}})R)}{(\varepsilon + \sqrt{\frac{s}{4}})^3 R^3} \geq \frac{1}{3} \frac{\pi^2 (1-\varepsilon) s}{(\varepsilon + \sqrt{\frac{s}{4}})^3}$$

for any sufficiently small $\varepsilon > 0$. Therefore the conclusion follows by taking the limit for $\varepsilon \to 0$. □

From Proposition 7.3 and 7.7, we obtain the followings.

Theorem 7.8. *Let $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{Z}}$ and $s > 0$. Then the volume growth of hyperkähler metric $g_\lambda^{(s)}$ is given by*

$$\lim_{r\to+\infty} \frac{V_{g_\lambda^{(s)}}(r)}{r^3} = \frac{8\pi^2}{3\sqrt{s}}.$$

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